NUMERICAL APPROXIMATION OF 1ST KIND VOLTERRA CONVOLUTION INTEGRAL EQUATIONS WITH DISCONTINUOUS KERNELS

PENNY J DAVIES AND DUGALD B DUNCAN

ABSTRACT. The cubic "convolution spline" method for first kind Volterra convolution integral equations was introduced in [*Convolution spline approximations of Volterra integral equations, J. Integral Equations Appl., 26:369–410, 2014*]. Here we analyse its stability and convergence for a broad class of piecewise smooth kernel functions and show it is stable and fourth order accurate even when the kernel function is discontinuous. Key tools include a new discrete Gronwall inequality which provides a stability bound when there are jumps in the kernel function, and a new error bound obtained from a particular B-spline quasi-interpolant.

1. Introduction. In **[5]** we derived a new numerical method which can be used to approximate the solution $u(t)$ of the first kind Volterra integral equation (VIE)

$$\int_0^t K(\tau)\, u(t-\tau)\, d\tau = a(t) \quad \text{for } t \in [0,T] \tag{1.1}$$

(where $a(0) = 0$ and $K(0) \neq 0$) with fourth order accuracy when the convolution kernel K and right-hand side a are sufficiently smooth. This "convolution spline" approximation shares some properties with Lubich's convolution quadrature **[11]**, but is explicitly constructed in terms of cubic spline basis functions. Although numerical results in **[5]** indicate that the scheme is also fourth order convergent when K is only piecewise smooth, the analysis does not extend to this case. We now provide a proof when $K(t)$ is piecewise smooth with (finite) jump discontinuities *irrespective of where the jumps occur*. In particular, convergence does **not** rely on fitting or adapting the stepsize so that the jumps occur at element boundaries, in contrast to the requirements of the trapezoidal rule (collocation with continuous piecewise linear approximation of u) applied to (1.1) with a step function kernel **[5,** §4.2.2] and methods for second kind problems in e.g. **[3,** Ch. 4.2] and **[13]**.

The discontinuous kernel convolution first kind VIEs we consider are also called VIEs with constant non-vanishing delays **[3,** Ch. 4]. These problems are sometimes written as Volterra functional equations where initial data specifying $u(t)$ in some initial interval are given. We do not consider the functional form here since it is equivalent to a problem in the form (1.1) after a shift in the time variable and absorbing the initial data into $a(t)$.

Much of the literature on discontinuous kernel problems for VIEs concentrates on problems of the second kind. One of the key early papers (from 1911) describing and analysing such second kind problems is **[8]** and recent numerical analysis for particular types of discontinuous second kind problems can be found in **[12, 13]**. Collocation methods for both first and second kind VIEs with discontinuous kernels are described by Brunner in **[3,** Sec. 4.2 & 4.3], and there is work on the analysis and numerical analysis of a different type of discontinuous kernel first kind problems in **[14, 18]**. That work is for problems with proportionate, vanishing delays and does not apply to our class of problems.

Convolution quadrature methods **[11, 1]** can also be used for discontinuous kernel problems in the form (1.1). However they rely on being able to evaluate the Laplace transform of $K(t)$, which is not always straightforward, and care may be needed to evaluate the contour integrals for the weights used in the scheme when there are jumps in K. Our method does not use the Laplace transform of the kernel K and the calculation of the weights is straightforward, with or without jumps.

Such discontinuous kernel problems arise in a variety of applications. Some first kind VIEs with a discontinuous kernel are derived in Laplace transform format in **[2, 17]**. They arise as part of a separation of variables solution of a scattering problem from a sphere in 3D and circle in 2D. For example, time-dependent acoustic scattering from a unit sphere can be decoupled into independent VIEs by expanding the incident wave into spherical harmonics, and in this case the nth order spherical harmonic modes of the surface potential satisfy (1.1) with kernel

$$K(t) = \frac{1}{2}\, P_n(1 - t^2/2)\, H(2-t)\,,$$

1991 AMS *Mathematics subject classification.* 65R20, 65M12.
Keywords and phrases. Volterra integral equations, discontinuous kernel, time delay.
Received by the editors June 20, 2016.

where $H(t)$ is the Heaviside function and $P_n(t)$ the degree n Legendre polynomial (see **[7]** for details).

Another important application area is in the deconvolution of well test data from water or oil reservoirs to obtain a constant rate drawdown response function that is then used to estimate important physical properties of the reservoir. One form of this problem is given in **[10,** Eq. 4.5]. In terms of (1.1), $u(t)$ is the unknown constant rate drawdown response, $K(t)$ is an actual or measured flow rate, and $a(t)$ a measured pressure change. An "ideal" well test experiment flows the well at a constant rate for a finite time and then closes the flow valve, continuing to measure the pressure change $a(t)$, so again $K(t)$ involves a Heaviside function. More realistic tests may involve switching the flow on and off a few times, or have a generally smooth flow rate $K(t)$ with a small number of jumps. It is also common for the measured flow rate data to be interpolated by piecewise constant or linear functions. More details can be found in e.g. **[4, 9]**.

In order to illustrate the solution structure of (1.1) when the kernel is discontinuous, we consider the kernel $K(t) = 1 - H(t - T_1)$, i.e. $K = 1$ for $0 < t < T_1$ and is zero otherwise. Taking the Laplace transform of (1.1), whose left-hand side is a Laplace convolution, using the notation $\bar{K}(s) = \mathcal{L}[K(t); s]$ gives $\bar{K}(s)\,\bar{u}(s) = \bar{a}(s)$, where $\bar{K}(s) = \left(1 - e^{-T_1 s}\right)/s$. Thus $(1 - e^{-sT_1})\bar{u}(s) = s\bar{a}(s)$, and taking the inverse transform gives the difference equation $u(t) - u(t - T_1) = a'(t)$, which has solution

$$u(t) = \sum_{k=0}^{\infty} a'(t - kT_1) = \sum_{k=0}^{\lfloor t/T_1 \rfloor} a'(t - kT_1)\,, \tag{1.2}$$

where $\lfloor b \rfloor = \text{floor}(b)$ is the largest integer less than or equal to b. If $a(t)$ has compact support in an interval $t \in [t_L, t_R]$ of width $t_R - t_L \le T_1$, then the solution (1.2) is T_1-periodic for all $t \ge t_L$. If a is localised in a region with rapid decay away from that region (but not compact support), the solution will be close to T_1-periodic. Note that the solution may also be obtained formally by writing the Laplace transform solution as

$$\bar{u}(s) = \frac{1}{1 - e^{-sT_1}}\; s\bar{a}(s) = \left(\sum_{j=0}^{\infty} e^{-sjT_1} \right) s\bar{a}(s)$$

and then inverting term by term. The exponentials are transforms of time shift operators and $s\bar{a}(s)$ is the transform of $a'(t)$ because $a(0) = 0$.

The plan for the rest of this article is as follows. In Section 2 we derive properties of the exact solution of (1.1) under various assumptions on the regularity of a and K and also briefly describe the convolution spline approximation scheme. Section 3 contains numerical convergence results for representative benchmark problems with discontinuous kernels. Some tools needed for stability analysis are introduced in Section 4, including a new discrete Gronwall inequality to deal with the step changes in the kernel, and we use them to establish stability of the scheme for a broad class of problems with piecewise smooth kernels. These stability results are a key step in the convergence analysis of the scheme in Section 5, and we derive a new error bound using a quasi-interpolant from the space of cubic B-splines.

2. Preliminaries.

2.1. Solution properties. We first determine the regularity of the solution u of (1.1) under various assumptions on a and K. Because $K(0) \neq 0$ we rescale the problem and will always assume that $K(0) = 1$. We consider two different types of function a:

$$\text{either} \qquad a \in C^{d+1}[0, T]\,,\ a(0) = 0\,; \tag{2.3}$$

$$\text{or} \qquad a \in C^{d+1}[0, T]\,,\ a^{(j)}(0) = 0 \quad \text{for } j = 0 : d + 1 \tag{2.4}$$

for $d \ge 0$ to be specified.

Lemma 2.1 (**[3,** Thm. 2.1.9]**).** *If $K(0) = 1$, $K \in C^{d+1}[0, T]$ and (2.3) holds for some $d \ge 0$, then the unique solution u of (1.1) satisfies $u \in C^d[0, T]$.*

We now show that the special nature of the convolution kernel allows the regularity requirement on K to be relaxed, provided (2.4) holds.

Lemma 2.2. *If $K(0) = 1$, $K \in C^1[0,T]$ and (2.4) holds for some $d \geq 0$, then the unique solution u of (1.1) satisfies $u \in C^d[0,T]$ and $u^{(p)}(0) = 0$ for $p = 0 : d$.*

Proof. The continuity of u when $d = 0$ is covered by Lemma 2.1. Rewriting (1.1) as

$$\int_0^t K(t-\tau)\,u(\tau)\,d\tau = a(t) \quad \text{for } t \in [0,T] \tag{2.5}$$

and differentiating gives

$$u(t) + \int_0^t K'(t-\tau)\,u(\tau)\,d\tau = a'(t)\,,$$

which yields $u(0) = a'(0) = 0$.
If $d = 1$ then consider the VIE

$$\int_0^\xi K(\tau)\,v(\xi-\tau)\,d\tau = a'(\xi)\,. \tag{2.6}$$

By above, the unique solution v of (2.6) is continuous with $v(0) = 0$. Integrating (2.6) over $(0,t)$ using $a(0) = 0$ gives

$$a(t) = \int_0^t \int_0^\xi K(\tau)\,v(\xi-\tau)\,d\tau\,d\xi = \int_0^t K(\tau)\int_0^{t-\tau} v(\xi)\,d\xi\,d\tau$$

and comparison with (1.1) (whose solution is unique) gives

$$u(t) = \int_0^t v(\xi)\,d\xi\,.$$

Hence $u \in C^1[0,T]$ and $u'(0) = v(0) = 0$. The result for $d \geq 2$ follows from repeating this argument d times. ☐

Note that the derivative conditions of (2.4) guarantee that the extension of u by zero to the negative real axis is in $C^d(-\infty,T]$. If they do not hold, then any numerical approximation of (1.1) needs to be 'corrected' as described for convolution quadrature in **[11,** Sec. 3] in order to attain optimal convergence.
The next result deals with the case that the kernel is piecewise smooth but discontinous.

Lemma 2.3. *Suppose that*

$$K(t) = \begin{cases} K_0(t)\,, & t < T_1 \\ K_1(t)\,, & t > T_1 \end{cases}$$

for some $T_1 \in (0,T)$, where $K_0(0) = 1$, $K_0 \in C^{d+1}[0,T_1]$, $K_1 \in C^{d+1}[T_1,T]$ and in general $K_0(T_1) \neq K_1(T_1)$. Then if (2.4) holds, the unique solution u of (1.1) satisfies $u \in C^d[0,T]$ and $u^{(p)}(0) = 0$ for $p = 0 : d$.

Proof. Applying Lemma 2.2 for $t < T_1$ gives $u \in C^d[0,T_1]$ and $u^{(p)}(0) = 0$ for $p = 0 : d$. It remains to show that the solution u extends to $[0,\, T]$ with no decrease in regularity, and we do this inductively, by showing that the regularity can successively be extended by intervals of length T_1.
Let $\widehat{K}_0(t) \in C^{d+1}[0,T]$ be a smooth extension of the function K_0 to $[0,T]$, and set $K_D(t) = \widehat{K}_0(t+T_1) - K_1(t+T_1)$, so $K_D \in C^{d+1}[0,T-T_1]$. As inductive hypothesis we assume that $u \in C^d[0,\, j\,T_1]$ for some $j \geq 1$, and we need to show that $u \in C^d[0,\, T_M]$, where $T_M = \min\{(j+1)\,T_1,\, T\}$. We rewrite (2.5) for $t \leq T_M$ as

$$\int_0^t \widehat{K}_0(t-\tau)u(\tau)d\tau = a(t) + \sigma(t-T_1)\,,$$

where

$$\sigma(t) = \begin{cases} 0\,, & t < 0 \\ \displaystyle\int_0^t K_D(t-\tau)\,u(\tau)\,d\tau\,, & t \in [0, T_M - T_1]\,. \end{cases}$$

By construction $\sigma \in C^{(d+1)}[0,T_M - T_1]$, $\sigma(0) = 0$ and $\widehat{K}_0(t) \in C^{d+1}[0,T]$, and so we only need to show that $\sigma^{(p)}(0) = 0$ for $p = 1 : d+1$ in order to apply Lemma 2.2 and deduce that $u \in C^d[0,T_M]$. The pth derivative

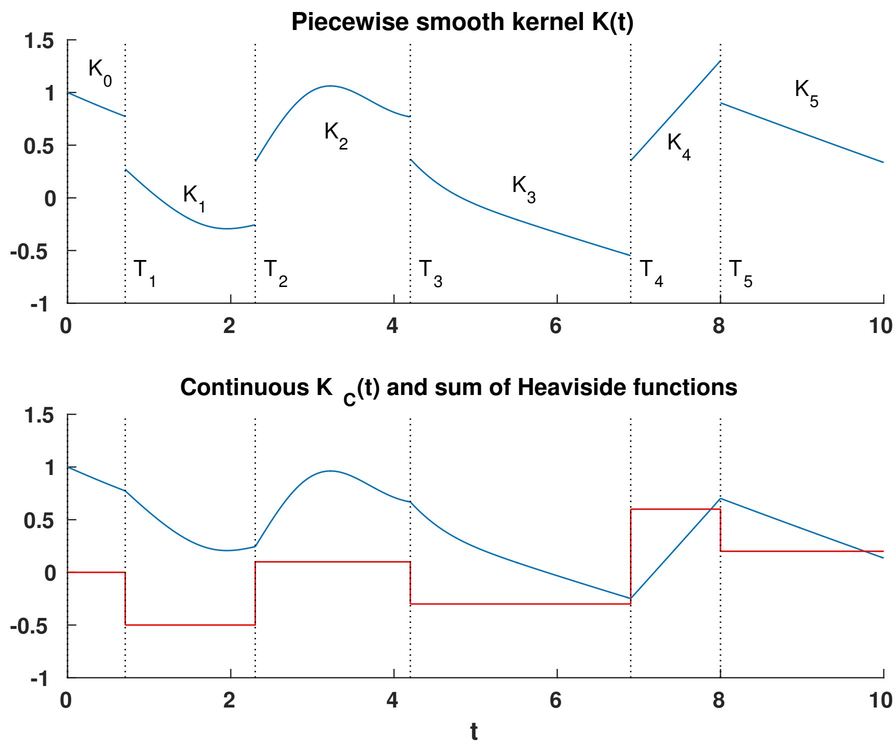

FIGURE 1. A piecewise smooth kernel function with $N_s = 5$ discontinuities.

of $\sigma(t)$ for $t \geq 0$ is

$$\sigma^{(p)}(t) = \sum_{j=0}^{p-1} K_D^{(j)}(0)\, u^{(p-1-j)}(t) + \int_0^t K_D^{(p)}(t-\tau)\, u(\tau)\, d\tau$$

from which the required result follows at $t = 0$. □

We allow the kernel K to have a finite number of discontinuities, at T_ℓ, $\ell = 1 : N_s$ where $0 = T_0 < T_1 < T_2 < \cdots < T_{N_s} < T_{N_s+1} = T$, and set $K_\ell(t) = K(t)$ for $t \in (T_\ell, T_{\ell+1})$. The arguments of Lemma 2.3 can be extended to this case, yielding the following result.

Corollary 2.1. *Suppose that a satisfies (2.4) and*

(2.7) $$K_0(0) = 1\,, \quad K_\ell \in C^{d+1}(T_\ell, T_{\ell+1}) \quad \text{for } \ell = 0 : N_s$$

for some $d \geq 0$. Then the unique solution u of (1.1) with $K(t) = K_\ell(t)$ for $t \in (T_\ell, T_{\ell+1})$, satisfies $u \in C^d[0,T]$ and $u^{(p)}(0) = 0$ for $p = 0 : d$.

Note that, as illustrated in Figure 1, a discontinuous kernel which satisfies (2.7) can be written as the sum of a continuous piecewise smooth function K_C and N_s constant pulse functions, i.e.

(2.8) $$K_C(t) := K(t) - \sum_{\ell=1}^{N_s} \alpha_\ell \left[H(t-T_\ell) - H(t-T_{\ell+1})\right]\,,$$

is continuous when

(2.9) $$\alpha_0 = 0 \quad \text{and} \quad \alpha_\ell - \alpha_{\ell-1} = K_\ell(T_\ell) - K_{\ell-1}(T_\ell)\,, \quad \ell = 1 : N_s\,.$$

Alternatively, (2.8) can be written as

$$K(t) = K_C(t) + \sum_{\ell=1}^{N_s} \left(K_\ell(T_\ell) - K_{\ell-1}(T_\ell)\right) H(t - T_\ell)\,.$$

2.2. Convolution spline approximation. The convolution spline scheme from **[5]** is a backwards-in-time approximation of the solution u of (1.1) at time $t_n = nh$ with constant stepsize $h = T/N_T$ given by

$$u(t_n - \tau) \approx U_n(t_n - \tau) = \sum_{j=0}^{n} v_{n-j}\,\phi_j(\tau/h) \quad \text{for } \tau \in [0, t_n], \tag{2.10}$$

where the basis functions are cubic B-splines with a parabolic runout condition at $t = 0$. That is, for $t \geq 0$,

$$\left.\begin{array}{ll} \phi_0(t) = B_3(t) + 3\,B_3(t+1)\,, & \phi_1(t) = B_3(t-1) - 3\,B_3(t+1)\,, \\ \phi_2(t) = B_3(t-2) + B_3(t+1)\,, & \phi_j(t) = B_3(t-j) \quad \text{for } j \geq 3\,, \end{array}\right\} \tag{2.11}$$

where $B_3(t)$ is the cardinal cubic B-spline (see e.g. a standard text such as **[6]**). All the basis functions ϕ_j are non-negative on $[0, \infty)$ except for ϕ_1, which is negative for $t \in [0, 1 - \sqrt{2/3})$. The cardinal B-spline $B_m(t)$ for $m \geq 1$ is a positive, even function, is globally C^{m-1}, has support in $(-(m+1)/2, (m+1)/2)$ and is a polynomial of degree m on each interval $(k, k+1)$ for $k = -(m+1)/2 : (m-1)/2$. It satisfies

$$B'_{m+1}(t) = B_m(t + 1/2) - B_m(t - 1/2)\,,$$

and integrating gives

$$B_{m+1}(x + 1/2) = \int_x^{x+1} B_m(t)\,dt \tag{2.12}$$

for $x > -(m+3)/2$.
Using the fact that $u(t) = 0$ for $t \leq 0$ (in other words, u is causal), (1.1) can be written as

$$\int_0^\infty K(\tau)\,u(t-\tau)\,d\tau = a(t), \quad t \in [0, T].$$

Substituting $t = t_n$ and the approximation (2.10) into this gives the discrete convolution equation

$$\int_0^\infty K(\tau)\,U_n(t_n - \tau)\,d\tau = \sum_{j=0}^{n} q_j\,v_{n-j} = a(t_n) \quad \text{for } n = 0 : N_T \tag{2.13}$$

for the unknown coefficients v_k, where

$$q_j = \int_0^\infty K(t)\,\phi_j(t/h)\,dt = h \int_{\max(0,j-2)}^{j+2} K(th)\,\phi_j(t)\,dt\,. \tag{2.14}$$

The v_k are obtained recursively from (2.13) by time marching:

$$v_0 = 0, \quad v_n = \frac{1}{q_0}\left(a(t_n) - \sum_{j=0}^{n-1} q_{n-j}\,v_j\right), \quad n \geq 1. \tag{2.15}$$

The step size $h := T/N_T$ is chosen independently of the locations T_ℓ of the jumps in $K(t)$. These locations are associated with mesh intervals by defining $m_\ell := \lfloor T_\ell/h \rfloor \in \mathbb{Z}$ and $r_\ell := T_\ell/h - m_\ell \in [0, 1)$ so that

$$T_\ell = (m_\ell + r_\ell)\,h \quad \text{for } \ell = 1 : N_s. \tag{2.16}$$

The case $r_\ell = 0$ only happens if the jump location is exactly at a mesh point, and in general $r_\ell > 0$. For completeness we set $m_0 = 0$, $m_{N_s+1} = N_T$ and $r_0 = r_{N_s+1} = 0$. We assume that step size h is sufficiently small so that successive T_ℓ do not occur in intervals which are near-neighbours, in particular we assume that

$$m_{\ell+1} - m_\ell \geq 5\,, \quad \ell = 0 : N_s \tag{2.17}$$

in the calculations below.

3. Benchmark problems and numerical results. Numerical results for the convolution spline approximation (2.10) of (1.1) for a unit step (i.e. $K(t) = 1 - H(t - T_1)$) are given in **[5]**, and we now examine the scheme's performance on some more complicated benchmark problems. Stability and covergence results for these classes of kernels are given in Sections 4–5.

3.1. BM1: discontinuous multiple step-function kernel. Suppose that K satisfying (2.7) is a piecewise constant function, i.e.

$$K(t) = \sum_{\ell=0}^{N_s} \alpha_\ell \left[H(t - T_\ell) - H(t - T_{\ell+1})\right] , \quad t \in [0, T] , \tag{3.18}$$

for some $\alpha_\ell \in \mathbb{R}$. This can be rearranged as

$$K(t) = 1 + \sum_{\ell=1}^{N_s} (\alpha_\ell - \alpha_{\ell-1}) \; H(t - T_\ell) , \quad \alpha_0 = 1 .$$

The exact solution of (1.1) with this kernel can again be obtained by Laplace transforms, using $\bar{K}(s) = (1 - \bar{Q}(s))/s$ where

$$\bar{Q}(s) = \sum_{\ell=1}^{N_s} (\alpha_{\ell-1} - \alpha_\ell) \, e^{-sT_\ell} .$$

The Laplace transform of the solution is obtained formally by writing

$$\bar{u}(s) = (1 - \bar{Q}(s))^{-1} \, s\bar{a}(s) = \sum_{j=0}^{\infty} \bar{Q}^j(s) \, s\bar{a}(s)$$

in the same way as for the single step kernel example in Section 1. The function $\bar{Q}(s)$ is the transform of a linear combination of time shift operators, with the property

$$\mathcal{L}^{-1}[\bar{Q}(s) \, s\bar{a}(s); t] = \mathsf{Q}a'(t) := \sum_{\ell=1}^{N_s} (\alpha_{\ell-1} - \alpha_\ell) \, a'(t - T_\ell) ,$$

giving

$$u(t) = \sum_{j=0}^{\lfloor t/T_1 \rfloor} \mathsf{Q}^j a'(t) .$$

Although messy to evaluate, it is possible to compute the exact solution up to any finite time given the causal nature of $a(t)$.

Assumption (2.17) implies that the first few coefficients q_j from (2.15) are

$$q_j/h = \begin{cases} 5/8 , & j = 0 \\ 5/6 , & j = 1 \\ 25/24 , & j = 2 . \end{cases}$$

For $j = m_{\ell-1} + 3 : m_\ell - 2$ the coefficients are $q_j = \alpha_{\ell-1} \, h$, and in the vicinity of the jump at T_ℓ they are

$$q_{m_\ell+k}/h = \alpha_{\ell-1} + (\alpha_\ell - \alpha_{\ell-1}) \int_{r_\ell - k}^{2} B_3(t) \, dt , \quad k = -1 : 2 .$$

3.2. BM2: piecewise smooth, globally C^0 (but not C^1) kernel. We consider the numerical test problem with kernel

$$K(t) = [1 - H(t - T_1)] \, \cos t$$

where $T_1 = \pi/2$. This has Laplace transform

$$\bar{K}(s) = \frac{s + e^{-T_1 s}}{1 + s^2}$$

and working through the formal Laplace transform procedure eventually gives the exact solution as

$$u(t) = \sum_{k=0}^{\lfloor t/T_1 \rfloor} (-1)^k \, \mathcal{I}^{k+1} \left[a(t - kT_1) + a''(t - kT_1)\right]$$

where $\mathcal{I}^k[f(t)]$ is the kth repeated integral of $f(t)$.

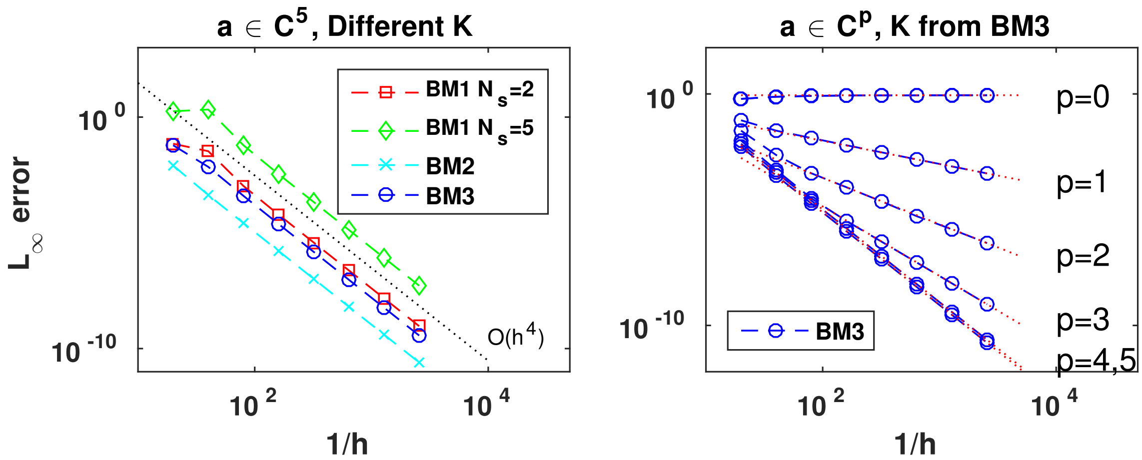

FIGURE 2. The left hand plot shows the L_∞ error for a range of different kernel functions from the benmark problem BM1–BM3 in subsections 3.1–3.3 with fixed $a \in C^5[0,T]$. The dotted line indicates the $\mathcal{O}(h^4)$ slope. The right hand plot has discontinuous kernel K given in subsection 3.3 and right hand side function $a \in C^p[0,T]$ for $p = 0 : 5$. The asymptotic slopes marked are $\mathcal{O}(h^p)$ for $p = 0 : 4$.

3.3. BM3: discontinuous kernel, not piecewise constant. Here we consider

$$K(t) = [1 - H(t - T_1)]\, e^{-t}$$

with Laplace transform

$$\bar{K}(s) = \frac{1 - e^{-T_1(1+s)}}{1+s}\,.$$

The Laplace transformed solution formally satisfies

$$\bar{u}(s) = \sum_{k=0}^{\infty} e^{-kT_1} e^{-skT_1}\,(1+s)\,\bar{a}(s)$$

giving the exact solution

$$u(t) = \sum_{k=0}^{\lfloor t/T_1 \rfloor} e^{-kT_1}\,\left(a(t-kT_1) + a'(t-kT_1)\right)\,. \tag{3.19}$$

Note that there is a decreasing contribution from terms further in the past.

3.4. Numerical implementation and results. Numerical results for the benchmark problems in the previous subsections are shown in Figure 2. In each case the coefficients q_j defined by (2.14) are evaluated almost exactly, using high order composite Gauss quadrature over intervals of length h between the nodes. If an interval contains one of the points of discontinuity T_ℓ for $\ell = 1 : N_s$ then it is split at the discontinuity and the same quadrature rule is applied on both segments. Errors in the solution are measured using the L_∞ norm of the difference between the exact and numerical solutions at the node points, when the exact solution is available. If not, then the error is estimated by mesh halving. In all cases the length of the interval is $T = 10$ and the step size h is chosen to avoid special cases in which the discontinuities occur at an integer multiple of h.

The plots on the left of Figure 2 all use the forcing term

$$a(t) = t^6 e^{-50(t-0.5)^2}, \quad t \geq 0$$

which satisfies (2.4) with $d = 4$. The BM1 (subsection 3.1) $N_s = 2$ case has

$$T_1 = 1/\sqrt{2},\; T_2 = \sqrt{3/2} \quad \text{with} \quad \alpha_0 = 1,\; \alpha_1 = 0.6,\; \alpha_2 = 0$$

while the BM1 $N_s = 5$ case has

$$T_1 = 1/\sqrt{2},\; T_2 = \sqrt{3/2},\; T_3 = \sqrt{5/2},\; T_4 = \sqrt{7/2},\; T_5 = \sqrt{11/2}$$

with

$$\alpha_0 = 1,\ \alpha_1 = 0.6,\ \alpha_2 = -0.4,\ \alpha_2 = -0.1,\ \alpha_4 = 0.5,\ \alpha_5 = 0.$$

Problem BM2 is as described in subsection 3.2 and problem BM3 from subsection 3.3 is used with $T_1 = 1/\sqrt{2}$. The scheme exhibits very clear $\mathcal{O}(h^4)$ convergence in all of these cases.

The results on the right of Figure 2 show what happens when the regularity of the forcing term $a(t)$ is reduced in problem BM3 with $T_1 = 1/\sqrt{2}$. We use

$$a(t) = \left(t^6 + (p+1)^2 (t-0.45)_+^{p+1}\right) e^{-50(t-0.5)^2}, \quad \text{for } p = 0:5,$$

where the truncated power function is $(x)_+ := \max(x, 0)$ for $x \in \mathbb{R}$. If $p = 0$ then $a \notin C^1[0,T]$ and the explicit solution u given by (3.19) is discontinuous at each integer multiple of T_1. Figure 2 shows that there is no convergence (in the L_∞ norm) when $p = 0$. If $p \geq 1$ then $a(t)$ satisfies (2.4) with $d = p-1$ and (3.19) gives $u \in C^{p-1}[0,T]$. The observed convergence rate is $\mathcal{O}(h^{\min(p,4)})$, saturating at $\mathcal{O}(h^4)$, which is better than might be expected for cubic spline interpolation, where $u \in C^4$ is a standard assumption. We note that the function u from (3.19) is smooth everywhere except at integer multiples of T_1 where its fourth derivative is discontinuous, and this special structure might be responsible for the better than expected convergence behaviour.

4. Stability of the convolution spline scheme. We now describe a new technique for investigating the stability (as defined below) of approximation schemes for (1.1). The advantage of this approach over that from **[5]** is that it enables us to prove convergence for discontinuous kernel functions.

Definition 4.1. *The approximation (2.15) of (1.1) is **stable** if there exists a constant C independent of h such that*

$$|v_n| \leq C \quad \text{for } n = 1 : N_T. \tag{4.20}$$

We first collect together some definitions and results which will be needed for the subsequent analysis.

4.1. Definitions and auxiliary results. We set $\|f\| = \|f\|_{L^\infty[0,T]}$ and define the broken norm $|\!|\!|\cdot|\!|\!|$ by

$$|\!|\!|f|\!|\!| := \sum_{\ell=0}^{N_s} \|f\|_{L^\infty(T_\ell, T_{\ell+1})},$$

where the points T_ℓ for $\ell = 1 : N_s$ are the permitted points of discontinuity of the kernel K. Note that (2.7) implies

$$|\!|\!|K|\!|\!| + |\!|\!|K'|\!|\!| \leq C$$

for some constant C.

Definition 4.2. *The Z-transform of a sequence $\{f_n\}_{n=0}^\infty$ is the function F given by*

$$F(\xi) = \mathcal{Z}\{f_n\}(\xi) = \sum_{n=0}^\infty f_n\, \xi^n \tag{4.21}$$

where $\xi \in \mathbb{C}$ with $|\xi| \leq 1$ is such that the sum converges.

The sequence μ_n defined by

$$15\,\mu_n + 5\,\mu_{n-1} + 5\,\mu_{n-2} - \mu_{n-3} = 0 \quad \text{for } n \geq 1 \text{ with } \mu_0 = 1 \text{ and } \mu_n = 0 \text{ for } n < 0, \tag{4.22}$$

plays a key part in the analysis, and its relevant properties are collected below.

Lemma 4.1. *The Z-transform of the sequence μ_n satisfies $\mathcal{Z}\{\mu_n\}(\xi) = 1/G_0(\xi)$ where*

$$G_0(\xi) := \left(15 + 5\,\xi + 5\,\xi^2 - \xi^3\right)/15 \tag{4.23}$$

has roots $\xi_1 \approx 6.197$, $\xi_{2,3} \approx -0.5986 \pm 1.4359\, i$. The solution of the difference equation (4.22) is

$$\mu_n = c_1\, \xi_1^{-n} + c_2\, \xi_2^{-n} + c_3\, \xi_3^{-n}, \tag{4.24}$$

where $c_1 \approx 0.050$, $\overline{c_3} = c_2 \approx 0.475 - 0.0897\,i$, *and*

$$C_\mu := \sum_{j=0}^{\infty} |\mu_j| \approx 2.051339\,. \tag{4.25}$$

We use the standard discrete Gronwall inequality below for continuous kernel problems.

Lemma 4.2 (Discrete Gronwall inequality; see e.g. **[16,** Lemma 1.4.2]). *If the sequence* $x_n \geq 0$ *satisfies*

$$x_0 \leq a\,, \quad x_n \leq a + b\sum_{j=0}^{n-1} x_j, \quad \text{for } n \geq 1$$

for some $a,\ b \geq 0$*, then*

$$x_n \leq a\,(1+b)^n \leq a\,e^{bn} \quad \text{for all } n \geq 0.$$

Discontinuous kernels whose first discontinuity is at $T_1 \approx Mh$ give rise to a stability sequence which has a localised contribution coming from M steps back. The following result extends the standard Gronwall inequality bound to deal with this case.

Lemma 4.3. *If the sequence* $x_n \geq 0$ *satisfies*

$$x_n \equiv 0 \text{ for } n < 0, \quad x_0 \leq a \quad \text{and} \quad x_n \leq a + b\sum_{j=0}^{n-1} x_j + c\,x_{n-M} \quad \text{for } n \geq 1 \tag{4.26}$$

with $a,\ b,\ c \geq 0$*, then*

$$x_n \leq a\,(1+b)^n (1+c)^{\lfloor n/M \rfloor} \quad \text{for all } n \geq 0, \tag{4.27}$$

where $\lfloor w \rfloor$ *is the largest integer less than or equal to* $w \in \mathbb{R}$*.*

Proof. We use induction over blocks of length M on the sequence $x_n \geq 0$ satisfying (4.26), with inductive hypothesis:
(IH)$_S$: (4.27) holds for $n = 0 : S\,M - 1$ for some $S \geq 1$.

It follows from Lemma 4.2 that (IH)$_S$ holds when $S = 1$, and suppose that it is true for some $S \geq 1$. We need to show that (4.27) holds for $n = SM + k$ for $k = 0 : M - 1$. For such k it follows from (4.26) and (4.27) that

$$\begin{aligned} x_{SM+k} &\leq a + a\,(1+c)^{S-1}\left\{c\,(1+b)^{(S-1)M+k} + b\sum_{j=0}^{SM-1}(1+b)^j\right\} + b\sum_{j=0}^{k-1} x_{SM+j} \\ &\leq a\,(1+c)^{S-1}\left\{c\,(1+b)^{(S-1)M+k} + (1+b)^{SM}\right\} + b\sum_{j=0}^{k-1} x_{SM+j} \\ &\leq a\,(1+c)^S(1+b)^{SM} + b\sum_{j=0}^{k-1} x_{SM+j}\,. \end{aligned}$$

We have thus shown that the sequence $y_k = x_{SM+k}$ satisfies

$$y_k \leq a\,(1+c)^S(1+b)^{SM} + b\sum_{j=0}^{k-1} y_j$$

and hence it follows from Lemma 4.2 that

$$x_{SM+k} \leq a\,(1+c)^S(1+b)^{SM}(1+b)^k = a\,(1+c)^S(1+b)^{SM+k}\,,$$

giving (IH)$_{S+1}$ as required. □

We also need the following weighted integral mean value theorem (see e.g. **[19,** Thm. A.6]).

Lemma 4.4. *If f is continuous on $[a,b]$, then for any non-negative weight function w with positive integral, there exists $\xi \in [a,b]$ such that*

$$f(\xi)\int_a^b w(x)\,dx = \int_a^b f(x)\,w(x)\,dx\,. \tag{4.28}$$

4.2. Stability for piecewise smooth kernels. In this subsection we start by taking the backward difference of the approximation (2.13) and obtain bounds on the sizes of the quantities $(q_j - q_{j-1})/q_0$ that appear, most of which are $\mathcal{O}(h)$. As noted in (2.8) the discontinuous kernel $K(t)$ can be written as the sum of a collection of Heaviside functions and a continuous piecewise smooth function and we establish the stability of these two cases separately in subsections 4.2.1 and 4.2.2. These two results are combined to give the general case in subsection 4.2.3.

We assume that (2.4) and (2.7) hold for some $d \geq 0$, and that h is small enough for (2.17) to hold, so that the first discontinuity of K occurs beyond the support of $\phi_j(t/h)$ for $j = 0:3$. It then follows from Lemma 4.4 that there is $\xi_0 \in (0,2)$ with

$$\frac{q_0}{h} = \int_0^2 K(th)\,\phi_0(t)\,dt = K(0)\int_0^2 \phi_0(t)\,dt + hK'(h\xi_0)\int_0^2 t\,\phi_0(t)\,dt = \frac{5}{8} + \frac{31h}{120}\,K'(h\xi_0)\,,$$

and hence $q_0 > 0$ for sufficiently small h. We similarly obtain

$$q_j/h = \begin{cases} 5/6 + (59/60)\,h\,K'(h\xi_1), \quad j=1 \\ 25/24 + (241/120)\,h\,K'(h\xi_2), \quad j=2 \\ 1 + 3\,h\,K'(h\xi_3), \quad j=3\,, \end{cases}$$

for some $\xi_j \in [\max(0, j-2), j+2]$.

Taking the backward difference of (2.13) and dividing by q_0 gives

$$v_0 = 0, \quad \sum_{j=0}^{n} \eta_j\,v_{n-j} = \left(a(t_n) - a(t_{n-1})\right)/q_0, \quad n \geq 1 \tag{4.29}$$

where

$$\eta_0 := 1\,, \quad \eta_j := (q_j - q_{j-1})\,/q_0\,, \quad j \geq 1\,. \tag{4.30}$$

It follows from (2.4) and the above calculations that

$$\frac{|a(t_n) - a(t_{n-1})|}{q_0} \leq \frac{\frac{8}{5}\,a_\Delta}{1 + \frac{31}{75}\,hK'(h\xi_0)}\,, \quad n = 1:N_T \tag{4.31}$$

where

$$a_\Delta := \max_{n \leq N_T}\left\{\frac{|a(t_n) - a(t_{n-1})|}{h}\right\} \tag{4.32}$$

and the leading coefficients in (4.29) are

$$\eta_1 = 1/3 + \eta_1^*, \quad \eta_2 = 1/3 + \eta_2^*, \quad \eta_3 = -1/15 + \eta_3^*\,,$$

where $|\eta_j^*| \leq 2\,h\max\{|K'(h\xi)| : \xi \in [0,5]\}$ for $j = 1:3$ when h is sufficiently small. It is also straightforward to verify that if $K(t)$ is continuous for $t \in [t_{j-3}, t_{j+2}]$ for $j \geq 4$, then

$$|q_j - q_{j-1}| \leq h^2\,\|K'\|$$

(and $|\eta_j| \leq (h^2/q_0)\,\|K'\| \leq 2\,h\,\|K'\|$ for sufficiently small h), but if K is discontinuous at $t = T_\ell \in [t_{j-3}, t_{j+2}]$, then η_j is an $\mathcal{O}(1)$ quantity.

4.2.1. Discontinuous multiple step-function kernel. Stability results for the case of a single jump step-function kernel were obtained in **[5]**, but the modified Gronwall Lemma 4.3 introduced above allows us to obtain a sharper result, as well as treating the more difficult case of multiple jumps.

If the kernel K is given by (3.18) with $\alpha_0 = 1$, then the coefficients η_j are

$$\eta_1 = \eta_2 = 1/3, \quad \eta_3 = -1/15, \quad \text{and} \quad \eta_j = 0 \quad \text{for } j = m_{\ell-1}+4 : m_\ell - 2$$

for each ℓ. The values around the jump discontinuity at T_ℓ are

$$\eta_{m_\ell+k} = \frac{8}{5}\,(\alpha_\ell - \alpha_{\ell-1}) \int_{r_\ell-k}^{r_\ell-k+1} B_3(t)\,dt = \frac{8}{5}\,(\alpha_\ell - \alpha_{\ell-1})\,\beta_k(r) \quad \text{for } k = -1:3,$$

using (2.12), where $\beta_k(r) = B_4(r_\ell - k + 1/2) \geq 0$. Substituting the values of η_j into (4.29) gives

$$\frac{15\,v_n + 5\,v_{n-1} + 5\,v_{n-2} - v_{n-3}}{15} = \frac{8}{5}\left\{\frac{a(t_n) - a(t_{n-1})}{h} + \sum_{\ell=1}^{N_s}(\alpha_{\ell-1} - \alpha_\ell)\sum_{k=-1}^{3}\beta_k\,v_{n-m_\ell-k}\right\}$$

with $v_0 = 0$ for $k \leq 0$. The Z-transform of this difference scheme is

$$G_0(\xi)\,V(\xi) = \frac{8}{5}\left\{\frac{(1-\xi)A(\xi)}{h} + \sum_{\ell=1}^{N_s}(\alpha_{\ell-1} - \alpha_\ell)\sum_{k=-1}^{3}\beta_k\,\xi^{m_\ell+k}\,V(\xi)\right\},$$

where G_0 is defined in (4.23). Using Lemma 4.1 and taking the inverse transform gives

$$v_n = \frac{8}{5}\sum_{j=0}^{n}\mu_{n-j}\left\{\frac{a(t_j) - a(t_{j-1})}{h} + \sum_{\ell=1}^{N_s}(\alpha_{\ell-1} - \alpha_\ell)\sum_{k=-1}^{3}\beta_k\,v_{j-m_\ell-k}\right\},$$

and it follows from (4.25) and (4.31) that

$$|v_n| \leq \frac{8}{5}C_\mu a_\Delta + \frac{8}{5}\sum_{j=0}^{n}|\mu_{n-j}|\sum_{\ell=1}^{N_s}|\alpha_{\ell-1} - \alpha_\ell|\sum_{k=-1}^{3}\beta_k(r)|v_{j-m_\ell-k}| \quad \text{for } n \geq 1. \tag{4.33}$$

To make further progress with this inequality we introduce the cumulative maximum modulus

$$z_n := \max_{0\leq j\leq n}|v_j|, \quad n > 0 \tag{4.34}$$

with $z_n = 0$ for all $n \leq 0$. Then the second term on the right-hand side of (4.33) can be bounded by:

$$\frac{8}{5}\sum_{j=0}^{n}|\mu_{n-j}|z_{n-m_1+1}\sum_{\ell=1}^{N_s}|\alpha_{\ell-1} - \alpha_\ell|\sum_{k=-1}^{3}\beta_k(r) = \frac{8}{5}\,z_{n-m_1+1}\sum_{j=0}^{n}|\mu_{n-j}|\sum_{\ell=1}^{N_s}|\alpha_{\ell-1} - \alpha_\ell|$$

since $\sum_{k=-1}^{3}\beta_k(r) = 1$ for all $r \in [0,1)$ from the properties of quartic splines. Hence

$$|v_n| \leq C_1 + C_2\,z_{n-m_1+1}$$

for each $n \geq 0$, where $C_1 = 8\,C_\mu\,a_\Delta/5$, a_Δ is defined in (4.32) and $C_2 = 8\,C_\mu\,\sum_{\ell=1}^{N_s}|\alpha_{\ell-1} - \alpha_\ell|/5$. If $0 \leq k \leq n$ then

$$|v_k| \leq C_1 + C_2\,z_{k-m_1+1} \leq C_1 + C_2\,z_{n-m_1+1}$$

and so

$$z_n \leq C_1 + C_2\,z_{n-m_1+1}\,.$$

Finally, applying the modified Gronwall inequality Lemma 4.3 gives the stability bound

$$|v_n| \leq z_n \leq C_1\,(1 + C_2)^{\lfloor n/(m_1-1)\rfloor}$$

for $n = 1 : N_T$. Note that $\lfloor n/(m_1 - 1)\rfloor \leq T/(T_1 - 2h)$ and so $|v_n|$ is bounded independently of h.

4.2.2. Continuous piecewise C^1 kernels. The convergence proof of **[5]** (which implies stability) needs a, $K \in C^7[0,T]$, but as shown below far less regularity is needed. We assume that K is globally continuous on $[0,T]$ and that a and K respectively satisfy (2.4) and (2.7) with $d = 0$.
The scheme (4.29) can be rewritten as

$$\frac{15\,v_n + 5\,v_{n-1} + 5\,v_{n-2} - v_{n-3}}{15} = \frac{a(t_n) - a(t_{n-1})}{q_0} - \sum_{j=0}^{n-1}\eta^*_{n-j}v_j$$

where $\eta^*_0 = 0$, η^*_j for $j = 1 : 3$ are as defined in Section 4.2 and $\eta^*_j = \eta_j$ for $j \geq 4$. The bounds from Section 4.2 give

$$|\eta^*_j| \leq 2\,h\,\|K'\|$$

for each j. As in the previous subsection, taking the Z-transform of the above difference scheme gives

$$G_0(\xi)\, V(\xi) = \frac{(1-\xi)A(\xi)}{q_0} - \mathcal{Z}\{\eta_n^*\}(\xi)\, V(\xi)$$

and we again use Lemma 4.1 and take the inverse transform to obtain

$$V(\xi) = \mathcal{Z}\{\mu_n\}(\xi) \left\{ \frac{(1-\xi)A(\xi)}{q_0} - \mathcal{Z}\{\eta_n^*\}(\xi)\, V(\xi) \right\}$$

and

$$v_n = \sum_{j=0}^{n} \mu_{n-j} \left(\frac{a(t_j) - a(t_{j-1})}{q_0} \right) - \sum_{j=0}^{n} \mu_{n-j} \sum_{k=0}^{j-1} \eta_{j-k}^* v_k, \quad n \geq 1\,.$$

It then follows from Lemma 4.1 and the bounds of Section 4.2 that

$$|v_n| \leq 2\, C_\mu a_\Delta + 9\, C_\mu\, \|K'\|\; h \sum_{j=0}^{n-1} |v_j|$$

for $n = 1 : N_T$. The standard Gronwall inequality in Lemma 4.2 then gives the stability result

$$|v_n| \leq 2\, C_\mu a_\Delta\, \exp\left(2\, C_\mu\, \|K'\|\; n\, h\right) \leq 2\, C_\mu a_\Delta\, \exp\left(2\, C_\mu\, \|K'\|\; T\right)$$

for $n = 1 : N_T$ where a_Δ is defined in (4.32).

4.2.3. General piecewise C^1 kernel. The results of the previous two subsections are now combined to prove the following result.

Theorem 4.1. *Suppose that (2.4) and (2.7) hold for $d = 0$. Then for sufficiently small h, the solution v_n of (4.29) satisfies*

$$|v_n| \leq C_1\, e^{C_2 T}\, (1 + C_3)^{\lfloor n/(m_1 - 1) \rfloor}\,, \tag{4.35}$$

for $n = 1 : N_T$, where

$$C_1 := 2\, C_\mu a_\Delta\,, \quad C_2 := 2\, C_\mu\, \|K'\|\,, \quad C_3 := 2\, C_\mu \sum_{\ell=1}^{N_s} |K_{\ell-1}(T_\ell) - K_\ell(T_\ell)|\,, \tag{4.36}$$

C_μ given by (4.25) and $a_\Delta \leq \|a'\|$ is defined in (4.32).

Proof. As in (2.8) we write K as the sum of a continuous piecewise C^1 function K_C and piecewise constant functions:

$$K(t) = K_C(t) + \sum_{\ell=1}^{N_s} \alpha_\ell\, \left[H(t - T_\ell) - H(t - T_{\ell+1})\right]$$

where the α_ℓ are as defined in (2.9). We use the results of the previous two subsections to split the coefficients η_j into two parts:

$$\eta_j = \eta_j^\dagger + \eta_j^*$$

where the $\eta_j^\dagger$ terms correspond to the piecewise constant parts (see Section 4.2.1) and are given by

$$\eta_0^\dagger = 1, \quad \eta_1^\dagger = \eta_2^\dagger = 1/3, \quad \eta_3^\dagger = -1/15, \quad \text{and} \quad \eta_j^\dagger = 0 \quad \text{for } j = m_{\ell-1} + 4 : m_\ell - 2$$

for each ℓ. The values around the jump discontinuity at T_ℓ are

$$\eta_{m_\ell + k}^\dagger = \frac{h\, (\alpha_\ell - \alpha_{\ell-1})}{q_0}\, \beta_k(r)\,. \quad \text{for } k = -1 : 3,$$

As in the previous subsection, the remainder terms η_j^* satisfy

$$\eta_0^* = 0\,, \quad |\eta_j^*| \leq 2\, h\, \|K'\|\,.$$

The scheme (4.29) can be thus be written as

$$\frac{15\,v_n + 5(v_{n-1}+v_{n-2}) - v_{n-3}}{15} = \frac{a(t_n)-a(t_{n-1})}{q_0} - \sum_{j=0}^{n-1} \eta^*_{n-j} v_j + \frac{h}{q_0}\sum_{\ell=1}^{N_s}(\alpha_{\ell-1}-\alpha_\ell)\sum_{k=-1}^{3}\beta_k\, v_{n-m_\ell-k}\,.$$

We again take the Z-transform, use Lemma 4.1 and take the inverse transform to obtain

$$v_n = \sum_{j=0}^{n}\mu_{n-j}\left(\frac{a(t_j)-a(t_{j-1})}{q_0}\right) - \sum_{j=0}^{n}\mu_{n-j}\sum_{k=0}^{j-1}\eta^*_{j-k}v_k + \frac{h}{q_0}\sum_{\ell=1}^{N_s}(\alpha_{\ell-1}-\alpha_\ell)\sum_{k=-1}^{3}\beta_k\, v_{j-m_\ell-k}$$

which gives the bound

$$|v_n| \le C_1 + C_2\, h\sum_{j=0}^{n-1}|v_j| + C_3\, z_{n-m_1+1}$$

for $n \ge 1$, where z_n is the cumulative maximum defined in (4.34) and the constants C_i are given by (4.36). Note that C_3 is obtained because $|\alpha_\ell - \alpha_{\ell-1}| = |K_\ell(T_\ell) - K_{\ell-1}(T_\ell)|$. If $k \le n$ then

$$|v_k| \le C_1 + C_2\, h\sum_{j=0}^{n-1} z_j + C_3\, z_{n-m_1+1}$$

giving

$$z_n \le C_1 + C_2\, h\sum_{j=0}^{n-1} z_j + C_3\, z_{n-m_1+1}\,.$$

Finally, we use the modified Gronwall lemma 4.3 to obtain

$$z_n \le C_1\,(1+C_2h)^n\,(1+C_3)^{\lfloor n/(m_1-1)\rfloor}\,,$$

giving (4.35) as required. □

5. Convergence.

We show below that under reasonable hypotheses and for a wide range of kernel functions the difference between the exact solution u of (1.1) and its convolution spline approximation $U_n(t)$ satisfies

$$|U_n(t) - u(t)| \le \begin{cases} Ch^4\,, & 0 \le t \le t_{n-1} \\ Ch^3\,, & t_{n-1} < t \le t_n \end{cases}$$

for $n = 1 : N_T$. This is achieved by introducing a quasi-interpolant $\widehat{U}(t)$ from the cubic B-spline space, and showing that it is within $\mathcal{O}(h^4)$ of the exact solution, and within $\mathcal{O}(h^4)$ of the approximate solution over most of the range.

For technical reasons we need $u(t) \in C^4[-2h, T+2h]$, and so we extend the definition of $K(t)$ and $a(t)$ for t up to $T+2h$. The maximum norm taken over the range $[0, T+2h]$ is denoted by an asterisk, i.e.

$$\|\cdot\|_* = \|\cdot\|_{L_\infty[0,T+2h]}\,.$$

5.1. A quasi-interpolant of $u(t)$.

We assume that $u \in C^4[0, T+2h]$ with $u^{(p)}(0) = 0$ for $p = 0:4$ (Lemmas 2.2 and 2.3 give sufficient conditions on a and K for this). The extension of u by zero to the negative real axis is in $C^4[-2h, T+2h]$, and $\left\|u^{(p)}\right\|_{L_\infty[-2h,T+2h]} = \left\|u^{(p)}\right\|_*$ for $p = 0:4$.

Following Powell [15, Ch. 20.4] we define the quasi-interpolant $\widehat{U}$ of u by

$$\widehat{U}(t) := \sum_{j=0}^{N_T+1}\hat{u}_j\, B_3(t/h - j)\,, \quad t \in \mathbb{R} \tag{5.37}$$

with coefficients

$$\hat{u}_j = \frac{4}{3}u(t_j) - \frac{1}{6}(u(t_{j-1}) + u(t_{j+1}))\,, \quad j = 0 : N_T+1. \tag{5.38}$$

The function $\widehat{U}(t)$ has compact support with

$$\widehat{U}(t) = 0\,, \quad t \notin (-2h, T+3h)$$

and its approximation error is given in the following lemma.

Lemma 5.1. *Given $u \in C^4[-2h, T+2h]$ with $u(t) \equiv 0$ for $t \leq 0$, then $\widehat{U}$ defined by (5.37) satisfies*

$$\left\|\widehat{U} - u\right\|_{L_\infty[-2h,T]} \leq \frac{35h^4}{1152}\left\|u^{(4)}\right\|_* .$$

Proof. This follows results in **[15,** Chs 20.4, 22.4] by rewriting $\widehat{U}(t)$ in each interval $t_j \leq t \leq t_{j+1}$ for $j = -2 : N_T - 1$ as

$$\widehat{U}(t_j + sh) = \sum_{k=-2}^{3} u(t_{j+k})\, b(s-k) \quad s \in [0,1]$$

where

$$b(s) := \left(8\,B_3(s) - B_3(s+1) - B_3(s-1)\right)/6\,. \tag{5.39}$$

Standard B-spline properties show that $b(s)$ has compact support in $(-3,3)$ and

$$\sum_{k=-\infty}^{\infty} k^m\, b(s-k) = s^m \quad \text{for } m = 0:3\,. \tag{5.40}$$

Fix $j \leq N_T - 1$ and $t = t_j + sh \in [t_j, t_{j+1}]$ and let $\mathsf{L}_j : C[-2h, T+2h] \to \mathbb{R}$ be the linear functional defined by

$$\mathsf{L}_j[f] = f(t_j + sh) - \sum_{k=-2}^{3} f(t_{j+k})\, b(s-k)\,.$$

Using (5.40) to verify that L_j annihilates cubic polynomials is straightforward and it follows from the Peano kernel theorem that

$$u(t_j + sh) - \widehat{U}(t_j + sh) = \int_{t_{j-2}}^{t_{j+3}} P_K(\theta, s)\, u^{(4)}(\theta)\, d\theta$$

where

$$P_K(\theta, s) := \frac{1}{3!}\left((t_j + sh - \theta)_+^3 - \sum_{k=-2}^{3} b(s-k)\,(t_{j+k} - \theta)_+^3\right)$$

and $(x)_+$ is the truncated power term from Section 3.4. By definition $P_K(\theta, s) = 0$ for $\theta \notin (t_{j-2}, t_{j+3})$, and it can be shown that $P_K(\theta, s) \geq 0$ for $\theta \in (t_{j-2}, t_{j+3})$, e.g. by considering each of the intervals $(t_j, t_j + sh)$, $(t_j + sh, t_{j+1})$ and (t_{j+k}, t_{j+k+1}) for $k = -2, -1, 1, 2$ separately. Hence the integral mean value theorem (Lemma 4.4) can be applied and

$$u(t_j + sh) - \widehat{U}(t_j + sh) = u^{(4)}(\zeta_j) \int_{t_{j-2}}^{t_{j+3}} P_K(\theta, s) d\theta = u^{(4)}(\zeta_j)\frac{h^4}{72}(2 + 3s^2 - 6s^3 + 3s^4)$$

for some $\zeta_j \in (t_{j-2}, t_{j+3})$. The polynomial in s is positive with maximum value $35/1152$ and so

$$|u(t_j + sh) - \widehat{U}(t_j + sh)| \leq \frac{35\,h^4}{1152}|u^{(4)}(\zeta_j)| \leq \frac{35h^4}{1152}\left\|u^{(4)}\right\|_*$$

and the result follows. □

5.2. The difference between the approximate solution and the quasi-interpolant. Because the exact solution $u(t)$ of (1.1) is zero for $t \leq 0$, (2.13) gives

$$\int_0^\infty K(t)\, u(t_n - t)\, dt = a(t_n) = \int_0^\infty K(t)\, U_n(t_n - t)\, dt$$

for $n = 1 : N_T$, and so

$$R_n^2 := \int_0^\infty K(t)\,\left(u(t_n - t) - \widehat{U}(t_n - t)\right)\, dt = \int_0^\infty K(t)\,\left(U_n(t_n - t) - \widehat{U}(t_n - t)\right)\, dt \tag{5.41}$$

for $n = 1 : N_T$. It follows from (2.10) and (5.37) that if $t \in [0, t_n]$ then

$$\begin{aligned} U_n(t_n - t) - \widehat{U}(t_n - t) &= \sum_{j=0}^{n} v_{n-j}\, \phi_j(t/h) - \sum_{j=-1}^{n} \hat{u}_{n-j}\, B_3(t/h - j) \\ &= \sum_{j=0}^{n} \varepsilon_{n-j}\, \phi_j(t/h) - (\hat{u}_{n+1} - 3\hat{u}_n + 3\hat{u}_{n-1} - \hat{u}_{n-2})\, B_3(t/h+1) \end{aligned} \tag{5.42}$$

where $\varepsilon_j := v_j - \hat{u}_j$ are the nodal errors. Substituting this into (5.41) then gives

$$\sum_{j=0}^{n} q_j\, \varepsilon_{n-j} = R_n^1 + R_n^2, \quad n = 1 : N_T \tag{5.43}$$

where R_n^2 is defined above and

$$R_n^1 := (\hat{u}_{n+1} - 3\hat{u}_n + 3\hat{u}_{n-1} - \hat{u}_{n-2}) \int_0^\infty K(t)\, B_3(t/h+1)\, dt\,. \tag{5.44}$$

The nodal error equation (5.43) has the same coefficients as the approximation scheme (2.13), $\sum_{j=0}^{n} q_j v_{n-j} = a(t_n)$, and thus we can apply Theorem 4.1 with $R_n^1 + R_n^2$ in place of $a(t_n)$ to obtain the following result.

Lemma 5.2. *Suppose that (2.7) holds for $d \geq 0$. Then if h is sufficiently small,*

$$\max_{0 \le j \le N_T} |\varepsilon_j| \le C_A \max_{1 \le n \le N_T} \frac{|R_n^1 - R_{n-1}^1 + R_n^2 - R_{n-1}^2|}{h}\,.$$

where $C_A := 2\, C_\mu\, e^{C_2 T}\, (1 + C_3)^{(1+T/T_1)}$ for constants C_μ, C_2 and C_3 as defined in Theorem 4.1.

We now show that if the exact solution u of (1.1) is sufficiently smooth, then the difference of the residuals is $\mathcal{O}(h^5)$.

Lemma 5.3. *Suppose that the kernel $K(t)$ and right-hand side $a(t)$ of (1.1) satisfy (2.7) and (2.4) respectively with $d = 4$ for $t \in [0, T+2h]$. Then if h is sufficiently small, the residuals R_n^1 and R_n^2 defined by (5.44) and (5.41) satisfy*

$$|R_n^1 - R_{n-1}^1| \le \frac{h^5}{12} \left\| u^{(4)} \right\|_* \tag{5.45}$$

$$|R_n^2 - R_{n-1}^2| \le C_B\, h^5 \left\| u^{(4)} \right\|_* \tag{5.46}$$

where $C_B = \frac{35}{1152} \left(T\, \|K'\| + \sum_{\ell=0}^{N_s} |K(T_\ell^-) - K(T_\ell^+)| + 2h\, \|K'\|_{L_\infty[T, T+2h]} \right)$.

Proof. It follows from the integral mean value theorem (4.28) that

$$\int_0^\infty K(t)\, B_3(t/h+1)\, dt = \frac{h}{6} \int_0^1 (1-s)^3 K(sh)\, ds = \frac{h}{24} K(h\xi)$$

for some $\xi \in (0,1)$, and taking the difference of R_n^1 defined in (5.44) then gives

$$R_n^1 - R_{n-1}^1 = \frac{hK(h\xi)}{24}\, (\hat{u}_{n+1} - 4\hat{u}_n + 6\hat{u}_{n-1} - 4\hat{u}_{n-2} + \hat{u}_{n-3})\,.$$

It was shown in Corollary 2.1 that the given hypotheses on K and a give $u \in C^4[-2h, T+2h]$, and any C^4 function f satisfies the identity

$$f(t_{n+2}) - 4f(t_{n+1}) + 6f(t_n) - 4f(t_{n-1}) + f(t_{n-2}) = h^4 \int_{-2}^{2} B_3(s)\, f^{(4)}(t_n + sh)\, ds$$

(see e.g. [**15,** Thm. 22.3]). Rearranging the definition (5.38) of $\hat{u}_n$ thus gives

$$\hat{u}_{n+1} - 4\hat{u}_n + 6\hat{u}_{n-1} - 4\hat{u}_{n-2} + \hat{u}_{n-3} = h^4 \int_{-3}^{3} b(s)\, u^{(4)}(t_{n-1} + sh)\, ds\,,$$

for $b(s)$ defined in (5.39), and so

$$R_n^1 - R_{n-1}^1 = \frac{h^5 K(h\xi)}{24} \int_{-3}^{3} b(s)\, u^{(4)}(t_{n-1} + sh)\, ds\,.$$

Because $b(s)$ takes both positive and negative values the integral mean value theorem cannot be used directly, but it can be used after taking the modulus. We have

$$\int_{-3}^{3} |b(s)| ds = \frac{4222 + 84 \times 18^{1/3} + 25 \times 18^{2/3}}{3993} = 1.15548\ldots$$

which gives the bound (5.45) for sufficiently small h (because $K(0) = 1$).

In order to bound $R_n^2 - R_{n-1}^2$ note that

$$R_n^2 = \int_{-2h}^{t_n} K(t_n - t) \left(u(t) - \widehat{U}(t) \right) dt\,,$$

taking into account the causality of the exact solution ($u(t) = 0$ for $t \le 0$) and the compact support of $\widehat{U}(t)$. Then

$$R_n^2 - R_{n-1}^2 = \int_{-2h}^{t_n} \left(K(t_n - t) - K(t_{n-1} - t)\right) (u(t) - \widehat{U}(t))\, dt\,,$$

where, for convenience, we extend $K(t)$ by zero for $t < 0$. Hence

$$\left|R_n^2 - R_{n-1}^2\right| \le \left\|\widehat{U} - u\right\|_{L_\infty[-2h,T]} \int_{-2h}^{t_n} |K(t_n - t) - K(t_{n-1} - t)| dt \le \frac{35 h^4}{1152} \left\|u^{(4)}\right\| \int_{0}^{t_{n+2}} |K(t) - K(t-h)|\, dt$$

using Lemma 5.1. The bound (5.46) then follows from

$$\begin{aligned} \int_{T_\ell}^{T_{\ell+1}} |K(t) - K(t-h)|\, dt &= \int_{T_\ell}^{T_\ell + h} |K(t) - K(t-h)|\, dt + \int_{T_\ell + h}^{T_{\ell+1}} |K(t) - K(t-h)|\, dt \\ &\le h\, |K(T_\ell^-) - K(T_\ell^+)| + h^2\, \|K'\| + (T_{\ell+1} - T_\ell - h)\; h\; \|K'\|\,. \end{aligned}$$

□

Combining these lemmas yields our final convergence result.

Theorem 5.1. *Suppose that K and a satisfy the hypotheses of Lemma 5.3. Then for sufficiently small h, for each $n = 1 : N_T$ the approximate solution $U_n(t)$ for $t \in [0, t_n]$ given by (2.13) satisfies*

$$|U_n(t) - u(t)| \le C_E \left\|u^{(4)}\right\|_* h^4 + C_F \left\|u^{(3)}\right\|_* B_3(t/h - n - 1)\, h^3\,, \tag{5.47}$$

where

$$C_E := \frac{5}{3}\, C_A \left\{ \frac{1}{12} + C_B \right\} + \frac{35}{1152}\,, \qquad C_F := \frac{516 + 11\sqrt{11}}{450}$$

for C_A, C_B as defined in Lemmas 5.2–5.3. That is

$$|U_n(t) - u(t)| \le \begin{cases} C_E \left\|u^{(4)}\right\|_* h^4\,, & 0 \le t \le t_{n-1} \\ \frac{1}{6}\, C_F \left\|u^{(3)}\right\|_* h^3 + \mathcal{O}(h^4)\,, & t_{n-1} < t \le t_n\,. \end{cases}$$

Proof. We prove the result by adding and subtracting the quasi-interpolant $\widehat{U}$. For $t \in [0, t_n]$,

$$\begin{aligned} |U_n(t_n - t) - u(t_n - t)| &\le \left|U_n(t_n - t) - \widehat{U}(t_n - t)\right| + \left|\widehat{U}(t_n - t) - u(t_n - t)\right| \\ &= \left|\sum_{j=0}^{n} \varepsilon_{n-j}\, \phi_j(t/h) - R_n\, B_3(t/h + 1)\right| + \left|\widehat{U}(t_n - t) - u(t_n - t)\right| \\ &\le \sum_{j=0}^{n} |\varepsilon_{n-j}|\, |\phi_j(t/h)| + |R_n|\, B_3(t/h + 1) + \left|\widehat{U}(t_n - t) - u(t_n - t)\right| \end{aligned}$$

using (5.42), where

$$R_n = \hat{u}_{n+1} - 3\hat{u}_n + 3\hat{u}_{n-1} - \hat{u}_{n-2}\,.$$

It remains to bound the three terms on the right hand side of this inequality. The bound for the third term is given by Lemma 5.1:

$$\left|\widehat{U}(t_n - t) - u(t_n - t)\right| \le \frac{35h^4}{1152}\left\|u^{(4)}\right\|_*$$

and the term $|\varepsilon_{n-j}|$ can be bounded using Lemmas 5.2 and 5.3:

$$\max_{0\le j\le N_T} |\varepsilon_j| \le C_A\left\{\frac{1}{12} + C_B\right\}\left\|u^{(4)}\right\|_* h^4\,.$$

All the basis functions ϕ_j are non-negative apart from $\phi_1(t)$, whose minimum value is $\phi_1(0) = -1/3$. Hence

$$\begin{aligned}
\sum_{j=0}^{n} |\varepsilon_{n-j}|\,|\phi_j(t/h)| &= (|\phi_1(t/h)| - \phi_1(t/h))\,|\varepsilon_{n-1}| + \sum_{j=0}^{n} |\varepsilon_{n-j}|\,\phi_j(t/h)\\
&\le \left(\frac{2}{3} + \sum_{j=0}^{n}\phi_j(t/h)\right)\max_{0\le j\le N_T}|\varepsilon_j|\\
&\le \frac{5\,h^4}{3}\,C_A\left\{\frac{1}{12} + C_B\right\}\left\|u^{(4)}\right\|_*\,.
\end{aligned}$$

The term R_n can be bounded in a similar way to $R^1_n - R^1_{n-1}$ in Lemma 5.3. The divided difference identity

$$f(t_{n+1}) - 3f(t_n) + 3f(t_{n-1}) - f(t_{n-2}) = h^3\int_{-3/2}^{3/2} B_2(s)\,f^{(3)}(t_{n-1/2} + sh)\,ds$$

in terms of the quadratic B-spline $B_2(s)$ gives

$$\hat{u}_{n+1} - 3\hat{u}_n + 3\hat{u}_{n-1} - \hat{u}_{n-2} = h^3\int_{-5/2}^{5/2} b_2(s)\,u^{(3)}(t_{n-1/2} + sh)\,ds$$

where $b_2(s) = (8\,B_2(s) - B_2(s-1) - B_2(s+1))\,/6$, and so

$$\begin{aligned}
|\hat{u}_{n+1} - 3\hat{u}_n + 3\hat{u}_{n-1} - \hat{u}_{n-2}| &\le h^3\int_{-5/2}^{5/2} |b_2(s)|\,|u^{(3)}(t_{n-1/2} + sh)|\,ds\\
&= h^3\,|u^{(3)}(\zeta_n)|\int_{-5/2}^{5/2}|b_2(s)|ds = h^3\,|u^{(3)}(\zeta_n)|\,\frac{516 + 11\sqrt{11}}{450}
\end{aligned}$$

for some $\zeta_n \in (t_{n-3}, t_{n+2})$. Combining these three terms gives the bound

$$|U_n(t_n - t) - u(t_n - t)| \le C_E\left\|u^{(4)}\right\|_* h^4 + C_F\left\|u^{(3)}\right\|_* B_3(t/h + 1)\,h^3$$

which yields (5.47). The final bound follows from noting that $B_3(t/h - n - 1) = 0$ for $t \le t_{n-1}$ and its maximum value for $t \in (t_{n-1}, t_n]$ is 1/6. $\square$

Note that to obtain an $\mathcal{O}(h^4)$ approximation over the whole range $t \in [0, T]$ where $T = N_T\,h$ just involves running the scheme for one extra step to $n = N_T + 1$.

6. Conclusions. The convolution spline scheme (2.13)–(2.14) is a fourth order accurate approximation of the VIE (1.1) for general piecewise smooth (continuous or discontinuous) kernels which is efficient and straightforward to implement. The weights q_j involve integrals of the kernel function multiplied by B-splines (or combinations of B-splines when near $t = 0$) – these can be evaluated to high accuracy by standard quadrature, and discontinuities in the kernel do not present any extra difficulties. This is not the case for methods such as convolution quadrature which rely on calculations in the Laplace domain.

Although much improved from **[5]**, the regularity assumptions needed for the proof of Theorem 5.1 may not be optimal – the method appears stable and fourth order accurate for an even broader range of discontinuous kernels and forcing terms $a(t)$ than discussed here.

The numerical experiments in **[5]** indicate that the the convolution spline method performs well for time domain boundary integral equations, and we are investigating whether the present analysis can be extended to these problems.

REFERENCES

1. L Banjai and Ch. Lubich. An error analysis of Runge-Kutta convolution quadrature. *BIT*, 51:483–496, 2011.

2. L Banjai and S Sauter. Rapid solution of the wave equation in unbounded domains. *SIAM J. Numer. Anal.*, 47:227249, 2008.

3. H Brunner. *Collocation Methods for Volterra Integral and Related Functional Equations*. Cambridge University Press, Cambridge, 2004.

4. G Da Prat. *Well Test Analysis for Fractured Reservoir Evaluation*. Elsevier, 1990.

5. P J Davies and D B Duncan. Convolution spline approximations of Volterra integral equations. *Journal of Integral Equations and Applications*, 26(3):369–410, 2014.

6. C De Boor. *A Practical Guide to Splines*. Springer-Verlag, 1978.

7. C L Epstein, L Greengard, and T Hagstrom. On the stability of time-domain integral equations for acoustic wave propagation. *Discrete and Continuous Dynamical Systems*, 36(8):4367–4382, 2016.

8. G C Evans. Volterra's integral equation of the second kind, with discontinuous kernel, second paper. *Transactions of the Americal Mathematical Society*, 12(4):429–472, 1911.

9. R N Horne. *Modern Well Test Analysis: A Computer-Aided Approach*. Petroway, second edition, 1995.

10. F J Kuchuk, M Onur, and F Hollaende. *Pressure Transient Formation and Well Testing: Convolution, Deconvolution and Nonlinear Estimation*. Elsevier, 2010.

11. Ch. Lubich. Convolution quadrature and discretized operational calculus. I. *Numerische Mathematik*, 52:129–145, 1988.

12. E Messina, E Russo, and A Vecchio. Convergence of solutions for two delays Volterra integral equations in the critical case. *Applied Mathematics Letters*, 23:1162–1165, 2010.

13. E Messina, E Russo, and A Vecchio. Comparing analytical and numerical solution of a nonlinear two-delay integral equations. *Mathematics and Computers in Simulation*, 81:1017–1026, 2011.

14. I Muftahov, A Tynda, and D Sidorov. Numerical solution of Volterra integral equations of the first kind with discontinuous kernels. *arXiv*, 1507.06484v1, 2015.

15. M J D Powell. *Approximation Theory and Methods*. Cambridge University Press, 1981.

16. A Quarteroni and A Valli. *Numerical Approximation of Partial Differential Equations*. Springer, 1997.

17. S Sauter and A Veit. Retarded boundary integral equations on the sphere: exact and numerical solution. *IMA Journal of Numerical Analysis*, 2013.

18. N A Sidorov and D N Sidorov. On the solvability of a class of Volterra operator equations of the first kind with piecewise continuous kernels. *Mathematical Notes*, 96:811–826, 2014.

19. E Süli and D Mayers. *An Introduction to Numerical Analysis*. Cambridge University Press, 2003.

Department of Mathematics and Statistics, University of Strathclyde, 26 Richmond St, Glasgow, G1 1XH, UK
Email address: **penny.davies@strath.ac.uk**

Maxwell Institute for Mathematical Sciences, Department of Mathematics, Heriot–Watt University, Edinburgh, EH14 4AS, UK
Email address: **D.B.Duncan@hw.ac.uk**